Bernstein's Expectation

Behrad Moniri
University of Pennsylvania
bemoniri@seas.upenn.edu

Abstract

In this note, we derive an upper bound on the expected value of a random variable satisfying the Bernstein inequality.

Theorem 1. *Consider a random variable Z that satisfies the following Bernstein-type inequality:*

$$\mathbb{P}(Z \geq t) \leq C \exp\Big(\frac{-t^2}{2\sigma^2 + 2bt}\Big). \tag{1}$$

We have

$$\mathbb{E}[Z] \leq 4b(1 + \log(C)) + 2\sigma\big(\sqrt{\pi} + \sqrt{\log(C)}\big). \tag{2}$$

Proof. If $\sigma^2 \leq bt$ then $\sigma^2 + bt \leq 2bt$. Otherwise $\sigma^2 + bt \leq 2\sigma^2$. We can rewrite the bound on Z in the following form:

$$\begin{aligned}
\mathbb{P}(Z \geq t) &\leq C \exp\Big(\frac{-t^2}{2\sigma^2 + 2bt}\Big) \\
&\leq C \max\Big[\min\Big(\exp\big(\frac{-t^2}{4bt}\big), 1\Big), \min\Big(\exp\big(\frac{-t^2}{4\sigma^2}\big), 1\Big)\Big] \\
&\leq C\Big[\min\Big(\exp\big(\frac{-t^2}{4bt}\big), 1\Big) + \min\Big(\exp\big(\frac{-t^2}{4\sigma^2}\big), 1\Big)\Big].
\end{aligned}$$

By writing the expected value as the integral of the tail probability function and bounding the tail probability, it is easy to derive a bound on $\mathbb{E}(Z)$.

$$\begin{aligned}
\mathbb{E}[Z] &= \int_0^\infty \mathbb{P}(Z \geq t)\, dt \\
&\leq \int_0^\infty \min\Big\{1, C\exp\big(\frac{-t}{4b}\big)\Big\}\, dt + \int_0^\infty \min\Big\{1, C\exp\big(\frac{-t^2}{4\sigma^2}\big)\Big\}\, dt \\
&= 4b(1 + \log(C)) + 2\sigma\big(\sqrt{\pi} + \sqrt{\log(C)}\big).
\end{aligned}$$

Where the last inequality follows from computing the integrals. ∎