Chernoff Tail Bound vs. a Moment-Based Bound

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Abstract

In this note, we compare Chernoff tail bound with a simple bound based on moments.

Theorem 1. *Suppose that $X \geq 0$ and that the moment generating function of X in an interval around zero. Given $\delta > 0$ and integer k, we have*

$$\inf_{k=0,1,2,\ldots} \frac{\mathbb{E}[X^k]}{\delta^k} \leq \inf_{\lambda>0} \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda\delta}}. \tag{1}$$

Proof. We will prove that $\forall \lambda > 0 : \inf_{k=0,1,2,\ldots} \frac{\mathbb{E}[X^k]}{\delta^k} \leq \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda\delta}}$. The theorem follows from this statement. Define $y_k := \frac{\mathbb{E}[X^k]}{\delta^k}$ and $z_k = \frac{1}{k!}\lambda^k\delta^k$. By Taylor's expansion

$$\frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda\delta}} = \frac{\sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k \mathbb{E}[X^k]}{\sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k\delta^k} = \frac{\sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k\delta^k y_k}{\sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k\delta^k} = \frac{\sum_{k=0}^{\infty} z_k y_k}{\sum_{k=0}^{\infty} z_k} \geq \inf_{k=0,1,2,\ldots} y_k, \tag{2}$$

which proves the theorem. □

Corollary 2. *We can write two different tail bounds for a positive variable X:*

$$\mathbb{P}[X \geq t] \leq \inf_{k=0,1,2,\ldots} \frac{\mathbb{E}[X^k]}{t^k} \tag{3}$$

and

$$\mathbb{P}[X \geq t] \leq \inf_{\lambda>0} \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda\delta}}. \tag{4}$$

Theorem (1) *states that the first bound is tighter than the second. However, the second bound is more popular, because of the ease of extending to to sum of i.i.d. random variables.*