Bounding the Variance of a Sub-Gaussian Random Variable

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Abstract

In this note, we will study the variance of sub-gaussian random variables.

Theorem 1. *Let X be a zero-mean, σ^2-subGaussian random variable.* $\mathrm{Var}[X] \leq \sigma^2$.

Proof. By the subGaussian property, for any $\lambda \in \mathbb{R}$ we have $\mathbb{E}[e^{\lambda Y}] \leq e^{\lambda^2\sigma^2/2}$. We can write

$$
\begin{aligned}
2e^{\frac{\lambda^2\sigma^2}{2}} &\geq \mathbb{E}[e^{\lambda Y}] + \mathbb{E}[e^{-\lambda Y}] && (1)\\
&= 2 + 2\sum_{k=1}^{\infty} \lambda^{2k}\frac{\mathbb{E}[Y^{2k}]}{(2k)!} && (2)\\
&\geq 2 + 2\lambda^2\frac{\mathbb{E}[Y^2]}{2}. && (3)
\end{aligned}
$$

Hence, for all $\lambda > 0$, we have $h(\lambda) = 1 + \frac{\lambda}{2}\mathbb{E}[Y^2] - e^{\lambda\sigma^2/2} \leq 0$. We have $h(0) = 0$, hence $h'(0)$ should be negative. Thus

$$
h'(\lambda) = \frac{1}{2}\mathbb{E}[Y^2] - \frac{\sigma^2}{2}e^{\lambda\sigma^2/2}\Big|_{\lambda=0} \leq 0 \implies \mathbb{E}[Y^2] \leq \sigma^2, \tag{4}
$$

which concludes the proof. □

Remark 2. *The equality $\mathbb{E}[Y^2] = \sigma^2$ might not hold even if σ is the smallest number satisfying $\mathbb{E}[e^{\lambda Y}] \leq e^{\lambda^2\sigma^2/2}$. To see this, let X be the following discrete random variable*

$$
X = \begin{cases} +1, & p \\ 0, & 1-2p \\ -1, & p \end{cases} \tag{5}
$$

It can easily be seen that $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] = 2p$. Assume that the equality holds with $\sigma^2 = \mathrm{Var}[X] = 2p$.

$$
\begin{aligned}
g(\lambda) &\triangleq \log\mathbb{E}[e^{\lambda X}] - \frac{\lambda^2\sigma^2}{2} - \lambda\mu \\
&= \log(pe^{\lambda} + pe^{-\lambda} + 1 - 2p) - \lambda^2 p \leq 0 \quad \forall\lambda \in \mathbb{R}.
\end{aligned}
$$

Now, let $p = 0.1$ and $\lambda = 1$, then

$$
\log(0.1e^1 + 0.1e^{-1} + 1 - 0.2) - 1.1 \approx 0.00311 \geq 0
$$

which is a contradiction.